

ON CERTAIN STATISTICAL PROPERTIES OF CONTINUED FRACTIONS WITH EVEN AND WITH ODD PARTIAL QUOTIENTS

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ABSTRACT. We prove results concerning the joint limiting distribution of the renewal time of denominators and consecutive digits of random irrational numbers in the case of continued fractions with even partial quotients, with odd partial quotients, and for Nakada's α -expansions.

1. INTRODUCTION

Let (a_n) (respectively, (q_n)) denote the sequence of digits (resp., denominators of the convergents) in the regular continued fraction (*RCF*) expansion of an irrational number. For each $R > 1$, consider the *renewal time* $n_R := \min\{n : q_n > R\}$, so that $q_{n_R-1} \leq R < q_{n_R}$. As a consequence of their renewal-type theorem for the natural extension of the Gauss map associated with regular continued fractions (*RCF*), Sinai and Ulcigrai proved the existence of the joint limiting distribution of $(\frac{q_{n_R-1}}{R}, \frac{R}{q_{n_R}}, a_{n_R-K}, \dots, a_{n_R+K})$ with K a fixed nonnegative integer [15], as $R \rightarrow \infty$. The classical Gauss-Kuzmin statistics give the probability of a random x in $[0, 1]$ having a prescribed string of digits in its continued fraction expansion at the n th position, for large n ; the joint limiting distribution studied in [15, 16] gives the probability of a random x in $[0, 1]$ having a prescribed string of digits in its continued fraction expansion at the first place where the denominator of the convergent is larger than R , for large R . The joint limiting distribution may therefore be considered an analogue of Gauss-Kuzmin statistics. Employing an abstract characterization of denominators of successive convergents in the regular continued fraction expansion $RCF(x)$ of x , Ustinov succeeded in explicitly computing this limiting distribution in the *RCF* case [16].

Sinai and Ulcigrai's result has been subsequently extended to the situation of continued fractions with even partial quotients (*ECF*) by Cellarosi [3]. The *ECF* limiting distribution was further used in the renormalization of theta sums—that is, replacing the theta sum $\sum e^{\pi i \omega n^2}$ with a theta sum of the type $\sum e^{-\pi i n^2 / \omega}$ modulo a rescaling, rotation, and small error term—as the map $\omega \rightarrow -\frac{\omega}{2}$ modulo 2 is closely related to the forward shift of even continued fractions. This has lead to some new results about the distribution of normalized theta sums and geometrical properties of their associated curlicues [4, 14].

This paper studies this type of limiting distributions in the case of three types of continued fractions: *ECF*, *OCF* (continued fractions with odd partial quotients), and *NCF $_{\alpha}$* (the Nakada α -expansions, which include *NICF*, or continued fraction to the nearest integer, as a special case). In the *ECF* case we provide a direct proof of the main result in [3] while making the limiting distribution explicit. The analogous problem is also solved in the *OCF*

case, for which no ergodic theoretical approach is known at this time. As in [16], the key tool is providing an abstract characterization for pairs of successive convergents in $ECF(x)$ and $OCF(x)$, which may be of independent interest. The OCF case is the most intricate, because the sequence of denominators of successive convergents in $OCF(x)$ is not necessarily increasing as in the RCF , ECF , or NCF_α cases. Finally we provide an explicit relation between the NCF_α limiting joint distribution and the distribution computed in [16].

Concretely, for a given type of continued fraction expansion (ECF , OCF , or NCF_α), consider the *renewal time*

$$n_R = \min\{n \in \mathbb{N} : q_n > R\} = \min\{n \in \mathbb{N} : q_{n-1} \leq R < q_n\}, \quad R > 1,$$

and the joint limiting distribution of $(\frac{q_{n_R-1}}{R}, \frac{R}{q_{n_R}}, \omega_{n_R-K}, \dots, \omega_{n_R+K})$ with $\omega_k = (a_k, e_k)$, for fixed K , as $R \rightarrow \infty$. Here again, ω_k denote the continued fraction digits and q_n denote the denominators of the convergents for a given type of CF expansion (see Section 2 for more details).

We will evaluate the Lebesgue measure $\mathcal{L}_{x_1, x_2, x_3, x_4}^{E/O, \pm}(R)$ of the set of numbers $x \in \Omega := [0, 1] \setminus \mathbb{Q}$ for which there exist $\frac{P}{Q}, \frac{P'}{Q'}$ successive convergents in $ECF(x)$ (respectively in $OCF(x)$) such that for given x_1, x_2, x_3, x_4 the following conditions are satisfied:

$$\frac{Q}{R} \leq x_1, \quad \frac{R}{Q'} \leq x_2, \quad \frac{Q}{Q'} \leq x_3, \quad \text{and} \quad (1.1)$$

$$0 \leq \frac{Q'x - P'}{-Qx + P} \leq x_4 \quad \text{respectively} \quad -x_4 \leq \frac{Q'x - P'}{-Qx + P} \leq 0. \quad (1.2)$$

In both ECF and OCF situations, we take $x_1, x_2, x_3, x_4 \in (0, 1]$.¹ In the OCF case, the ratio of successive denominators Q/Q' can in fact be any rational number in the interval $(0, G)$, but since in the definition of n_R we are interested only in $Q \leq R < Q'$, we can restrict our attention to $x_3 \leq 1$ in the definition of $\mathcal{L}^{O, \pm}$. The golden ratios $G = \frac{1+\sqrt{5}}{2}$ and $g = \frac{1}{G} = \frac{-1+\sqrt{5}}{2}$ will be used often.

The terms $\frac{q_{n_R-1}}{R}$ and $\frac{R}{q_{n_R}}$ in the joint limiting distribution clearly relate to the parameters x_1 and x_2 in the function \mathcal{L} . Likewise, the digits ω_k in the joint limiting distribution relate to the parameters x_3 and x_4 in \mathcal{L} due to equalities (2.4) and (2.6).

The main result of this paper shows that $\mathcal{L}^{E/O, \pm}(R)$ has an explicitly computable limiting distribution as $R \rightarrow \infty$.

Theorem 1.1. *The joint distributions $\mathcal{L}_{x_1, x_2, x_3, x_4}^{E/O, \pm}(R)$ exist as $R \rightarrow \infty$ and*

$$\mathcal{L}_{x_1, x_2, x_3, x_4}^{E, \pm}(R) = \frac{2F_{\pm}}{3\zeta(2)} + O_{\varepsilon}(R^{-1+\varepsilon}), \quad (1.3)$$

$$\mathcal{L}_{x_1, x_2, x_3, x_4}^{O, +}(R) = \frac{F_+ - D_1}{\zeta(2)} + O_{\varepsilon}(R^{-1/2+\varepsilon}), \quad (1.4)$$

$$\mathcal{L}_{x_1, x_2, x_3, x_4}^{O, -}(R) = \frac{F_- - D_2 - D_3}{\zeta(2)} + O_{\varepsilon}(R^{-1/2+\varepsilon}),$$

¹If any of the parameters equals 0, then \mathcal{L} equals 0 as well, so we ignore this degenerate case.

where $F_{\pm} = F_{\pm}(x_1, x_2, x_3, x_4)$ and $D_i = D_i(x_1, x_2, x_3, x_4)$ are given by²

$$F_{\pm} = \mp \begin{cases} \text{Li}_2(\mp x_1 x_2 x_4) & \text{if } x_3 \geq x_1 x_2, \\ \text{Li}_2(\mp x_3 x_4) - \log(1 \pm x_3 x_4) \log \frac{x_1 x_2}{x_3} & \text{if } x_3 < x_1 x_2, \end{cases} \quad (1.5)$$

$$\begin{aligned} D_2 &= F_{-}(x_1, x_2, x_3, x_4) - F_{-}(x_1, x_2, \min\{x_3, g^2\}, x_4), \\ D_1 &= \sum_{\ell \geq 1} I_{\ell}^{+}, \quad D_3 = \sum_{\ell \geq 2} I_{\ell}^{-}, \quad \text{with} \end{aligned} \quad (1.6)$$

$$I_{\ell}^{\pm} = \int_{1/x_2}^{A_{\ell}} dx \int_{x/(2\ell+g)}^{B_{\ell}(x)} \frac{x_4 dy}{y(y \pm x_4 x)}, \quad \text{where} \quad (1.7)$$

$$A_{\ell} = (2\ell + g)x_1, \quad B_{\ell}(x) = B_{\ell, x_2, x_3}(x) = \min \left\{ x_3 x, x_1, \frac{x}{2\ell}, \frac{x-1}{2\ell-1} \right\}.$$

The integrals I_{ℓ}^{\pm} can be written explicitly as a combination of logarithms and dilogarithms.

Kraaikamp's metric theory for S -expansions [6] provides immediate characterizations of pairs of successive convergents for such continued fractions, which are obtained from RCF only by singularization (see the remark at the end of Section 3 for definition of singularization). In the last section we show how to compute the joint limiting distribution associated as above with Nakada's α -expansions [10] for $\frac{1}{2} \leq \alpha \leq 1$. The cases $\alpha = 1$ and $\alpha = \frac{1}{2}$ are best known, corresponding to the RCF and $NICF$ (continued fraction to the nearest integer). The latter was introduced by Minnigerode [9] and was also studied in [1, 13, 18]. Our calculations show explicit connections with Ustinov's RCF distribution.

2. BASIC ECF AND OCF PROPERTIES

For each $x \in \Omega$, the ECF (respectively, OCF) expansion of x is given by

$$x = \frac{1}{a_1 + \frac{e_1}{a_2 + \frac{e_2}{a_3 + \frac{e_3}{\ddots}}}} = [[(a_1, e_1), (a_2, e_2), (a_3, e_3), \dots]], \quad (2.1)$$

where $e_n \in \{\pm 1\}$ and all a_n 's are even positive integers (respectively, all a_n 's are odd positive integers with $a_n + e_n \geq 2$). For more details see [5, 6, 8, 11, 12, 13]. As in [5, 8], consider the "flipped" continued fraction map $T_D : [0, 1] \rightarrow [0, 1]$ for a subset D of $[0, 1]$, defined by $T_D(0) = 0$, $T_D(1) = 1$, and

$$T_D(x) = \begin{cases} \{1/x\} & \text{if } x \in (0, 1) \setminus D, \\ 1 - \{1/x\} & \text{if } x \in D, \end{cases}$$

with auxiliary functions

$$e_D(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \setminus D, \\ -1 & \text{if } x \in D, \end{cases} \quad a_D(x) = \begin{cases} [1/x] & \text{if } x \in [0, 1] \setminus D, \\ 1 + [1/x] & \text{if } x \in D. \end{cases}$$

²In this paper the convention is that $\int_a^b = 0$ when $a \geq b$.

Note that

$$T_D(x) = e_D(x) \left(\frac{1}{x} - a_D(x) \right), \quad \forall x \in (0, 1).$$

Consider the sets

$$D_O := \bigcup_{n \in 2\mathbb{N}} \left[\frac{1}{n+1}, \frac{1}{n} \right), \quad D_E := [0, 1) \setminus D_O = \bigcup_{n \in 2\mathbb{N}-1} \left[\frac{1}{n+1}, \frac{1}{n} \right).$$

Denote $D = D_E$ in the *ECF* case, respectively $D = D_O$ in the *OCF* case. In both *ECF* or *OCF* situations the signs $e_n = e_n(x)$ and the digits $a_n = a_n(x)$ are given, for $x \in \Omega$, by

$$e_0 = 1, \quad e_n = e_D(t_{n-1}), \quad a_0 = 0, \quad a_n = a_D(t_{n-1}),$$

where $t_n = t_n(x) = T_D^n(x)$. On the D -continued fraction expansion the iterates of the Gauss type map T_D act as a shift map by

$$T_D^n([(a_1, e_1), (a_2, e_2), \dots]) = [(a_{n+1}, e_{n+1}), (a_{n+2}, e_{n+2}), \dots], \quad \forall n \in \mathbb{N}_0.$$

The D -convergents $\frac{p_n}{q_n}$ are defined by

$$\begin{cases} p_{-1} = 1, & p_0 = 0, & p_n = a_n p_{n-1} + e_{n-1} p_{n-2}, \\ q_{-1} = 0, & q_0 = 1, & q_n = a_n q_{n-1} + e_{n-2} q_{n-2}, \end{cases} \quad (2.2)$$

or in equivalent formulation

$$\begin{aligned} \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} &= \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix} \begin{pmatrix} 0 & e_{n-1} \\ 1 & a_n \end{pmatrix} = \dots \\ &= \begin{pmatrix} 0 & e_0 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & e_1 \\ 1 & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 & e_{n-1} \\ 1 & a_n \end{pmatrix}, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (2.3)$$

The following elementary fundamental relations are satisfied:

$$\begin{aligned} p_{n-1} q_n - p_n q_{n-1} &= (-1)^k e_0 e_1 \dots e_{n-1} =: \delta_n, \\ \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} &= \frac{\delta_n}{q_{n-1} q_n}, \quad \forall n \in \mathbb{N}_0, \\ x &= \frac{p_n + p_{n-1} e_n t_n}{q_n + q_{n-1} e_n t_n}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

The latter equation is equivalent to

$$e_n t_n = e_n T_D^n(x) = \frac{q_n x - p_n}{-q_{n-1} x + p_{n-1}}, \quad \forall n \in \mathbb{N}. \quad (2.4)$$

Upon (2.4) we infer

$$0 < \left| \frac{q_n x - p_n}{-q_{n-1} x + p_{n-1}} \right| < 1, \quad \forall x \in \Omega, \quad \forall n \in \mathbb{N}. \quad (2.5)$$

It is well-known and plain to check for every continued fraction that if x is as in (2.1), then

$$\frac{q_{n-1}}{q_n} = [[(a_n, e_{n-1}), (a_{n-1}, e_{n-2}), \dots, (a_2, e_1), (a_1, *)]], \quad \forall n \in \mathbb{N}, \quad (2.6)$$

where $(a_1, *)$ means that the finite expansion terminates with a_1 .

3. SUCCESSIVE *ECF* AND *OCF* CONVERGENTS

In $GL_2(\mathbb{Z})$ consider the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and denote their images in $SL_2(\mathbb{Z}/2\mathbb{Z})$ by $[I], [J], [A], [B]$. Clearly $\{[I], [J]\}$ forms a subgroup on two elements of $SL_2(\mathbb{Z}/2\mathbb{Z})$ and $\{[I], [A], [B]\}$ forms a subgroup on three elements of $SL_2(\mathbb{Z}/2\mathbb{Z})$. Consider the sets

$$\mathcal{R} = \left\{ \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} : 0 \leq P \leq Q, 1 \leq P' \leq Q', \right\},$$

$$\mathcal{R}_E := \{M \in \mathcal{R} : 1 \leq Q \leq Q', M \equiv I \text{ or } J \pmod{2}\},$$

$$\mathcal{R}_O := \{M \in \mathcal{R} : \lambda_M > g, M \equiv I, A, \text{ or } B \pmod{2}\}.$$

For $M \in GL_2(\mathbb{Z})$ denote

$$\lambda_M = \frac{Q'}{Q}, \quad E_M(x) = \frac{Q'x - P'}{-Qx + P}, \quad x \notin \mathbb{Q}. \quad (3.1)$$

 3.1. Successive convergents for *ECF*(x).

Lemma 3.1. *In the ECF expansion, $q_k \geq q_{k-1} \geq 1$, $p_{k+1} \geq p_k \geq 1$, and $q_k - p_k \geq q_{k-1} - p_{k-1} \geq 1$ for every $k \geq 1$.*

Proof. Let (x_n) be a sequence defined by $x_n = a_n x_{n-1} + e_{n-1} x_{n-2}$ with a_n an even positive integers and $e_n \in \{\pm 1\}$. Suppose that $x_{k_0} \geq x_{k_0-1} \geq 1$ for some $k_0 \geq 1$. Then $x_{k_0+1} \geq 2x_{k_0} - x_{k_0-1} \geq x_{k_0}$. This shows inductively that $x_n \geq x_{n-1} \geq 1$ for every $n \geq k_0$. The statement follows by taking $(x_n, k_0) = (q_n, 1)$, $(x_n, k_0) = (p_n, 2)$, and respectively $(x_n, k_0) = (q_n - p_n, 1)$. \square

Furthermore, since $p_{n-1}q_n - p_nq_{n-1} = \pm 1$, it follows that $q_n(x) > q_{n-1}(x)$, for all $n \geq 2$ and $x \in \Omega$.

Proposition 3.2. *For each $x \in \Omega$ the following are equivalent:*

- (i) $\frac{P}{Q}, \frac{P'}{Q'}$ successive convergents in *ECF*(x).
- (ii) $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}_E$ and $0 < |E_M(x)| < 1$.

Proof. (i) \implies (ii) Suppose $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$ for some $n \geq 1$. From Lemma 3.1, $\begin{pmatrix} 0 & e_{k-1} \\ 1 & a_k \end{pmatrix} \equiv J \pmod{2}$ and equality (2.3) we infer that $M \in \mathcal{R}_E$. The second condition in (ii) follows from (2.5).

(ii) \implies (i) Consider first the case $Q = 1$. Only the matrices $M = \begin{pmatrix} 0 & 1 \\ 1 & Q' \end{pmatrix}$ and $M = \begin{pmatrix} 1 & Q'^{-1} \\ 1 & Q' \end{pmatrix}$ may arise. Since $M \equiv I \text{ or } J \pmod{2}$ only the former case can occur and Q' is necessarily an even positive integer. The corresponding inequality $0 < \left| \frac{Q'x-1}{-x} \right| < 1$ is equivalent to $x \in \left(\frac{1}{Q'+1}, \frac{1}{Q'} \right) \cup \left(\frac{1}{Q'}, \frac{1}{Q'-1} \right)$ or, according to the definition of a_1 , to $a_1 = Q'$, showing that $\frac{0}{1}, \frac{1}{Q'}$ are successive convergents of x .

When $Q > 1$, take ($\ell \geq 1$):

$$\begin{aligned} e_M &= 1, \quad Q_0 = Q' - 2\ell Q, \quad P_0 = P' - 2\ell P \quad \text{if } [\lambda] = 2\ell, \\ e_M &= -1, \quad Q_0 = 2\ell Q - Q', \quad P_0 = 2\ell P - P' \quad \text{if } [\lambda] = 2\ell - 1, \\ M_0 &= \begin{pmatrix} P_0 & P \\ Q_0 & Q \end{pmatrix}. \end{aligned}$$

In both cases one has $0 < Q_0 < Q$, $M = M_0 \begin{pmatrix} 0 & e_M \\ 1 & 2\ell \end{pmatrix}$, and so $M_0 \equiv I$ or $J \pmod{2}$. Since $Q' > Q > Q_0$, the condition $0 < |E_M| < 1$ is equivalent with x lying between $\frac{P'+P}{Q'+Q}$ and $\frac{P'-P}{Q'-Q}$, while $0 < |E_{M_0}| < 1$ is equivalent with x lying between $\frac{P+P_0}{Q+Q_0}$ and $\frac{P-P_0}{Q-Q_0}$. When $\frac{P}{Q} < \frac{P'}{Q'}$ the former implies the latter because of

$$\begin{aligned} \frac{P-P_0}{Q-Q_0} &= \frac{(2\ell+1)P-P'}{(2\ell+1)Q-Q'} < \frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} < \frac{P'-P}{Q'-Q} \\ &\leq \frac{P+P_0}{Q+Q_0} = \frac{P'-(2\ell-1)P}{Q'-(2\ell-1)Q} < \frac{P_0}{Q_0} = \frac{P'-2\ell P}{Q'-2\ell Q} \quad \text{when } [\lambda] = 2\ell, \end{aligned}$$

and of

$$\begin{aligned} \frac{P_0}{Q_0} &= \frac{2\ell P - P'}{2\ell Q - Q'} < \frac{P+P_0}{Q+Q_0} = \frac{(2\ell+1)P-P'}{(2\ell+1)Q-Q'} < \frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} \\ &< \frac{P'-P}{Q'-Q} \leq \frac{P-P_0}{Q-Q_0} = \frac{P'-(2\ell-1)P}{Q'-(2\ell-1)Q} \quad \text{when } [\lambda] = 2\ell-1. \end{aligned}$$

When $\frac{P'}{Q'} < \frac{P}{Q}$, analogous inequalities show that $0 < |E_M| < 1$ implies $0 < |E_{M_0}| < 1$. Furthermore, the inequalities $0 \leq P_0 \leq P$ follow from $|P'Q - PQ'| = |PQ_0 - P_0Q| = 1$ and $P \geq 1$. \square

3.2. Successive convergents for $OCF(x)$. Denominators of successive convergents for $OCF(x)$ satisfy ([11, Eq. 2.10])

$$\begin{aligned} r_n &:= \frac{q_n}{q_{n-1}} \\ &= a_n + e_{n-1} [[(a_{n-1}, e_{n-2}), (a_{n-2}, e_{n-3}), \dots, (a_2, e_1), (a_1, *)]] \\ &\geq a_n - [[(3, -1), (3, -1), \dots, (3, -1), (3, *)]] \\ &> a_n - [[(3, -1), (3, -1), (3, -1) \dots]] \\ &= a_n - 1 + 1/G = a_n - 2 + G. \end{aligned} \tag{3.2}$$

In the opposite direction one has

$$r_n = a_n + \frac{e_{n-1}}{r_{n-1}} < a_n + \frac{e_{n-1}}{a_{n-1} - 2 + G} \leq a_n + \frac{1}{G-1} = a_n + G. \tag{3.3}$$

In particular (3.2) and (3.3) show that if $a_n \geq 3$, then $r_n > 1 + G$, proving

Lemma 3.3. *If $r_n \leq 2 + g$ then $a_n = 1$, and in particular $e_n = 1$ and $0 < \frac{q_n x - p_n}{-q_{n-1} x + p_{n-1}} < 1$.*

Proposition 3.4. *For each $x \in \Omega$ the following are equivalent:*

- (i) $\frac{P}{Q}, \frac{P'}{Q'}$ successive convergents in $OCF(x)$.
- (ii) $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}_O$ and one of the following two conditions holds:
 - (*) $\lambda_M := \frac{Q'}{Q} > 2 + g$ and $0 < |E_M(x)| < 1$.

(**) $g < \lambda_M \leq 2 + g$ and $0 < E_M(x) < 1$.

Proof. (i) \implies (ii) Suppose that there is $n \geq 1$ such that

$$M = \begin{pmatrix} 0 & e_0 = 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & e_1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & e_{n-1} \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}. \quad (3.4)$$

Since $\begin{pmatrix} 0 & e_{i-1} \\ 1 & a_i \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = A \pmod{2}$ and $\{[I], [A], [B]\}$ forms a subgroup of $SL_2(\mathbb{Z}/2\mathbb{Z})$, it follows that $M \equiv I, A$, or $B \pmod{2}$. The inequality $GQ' > Q$ follows from (3.2), while $0 \leq P = p_{n-1} \leq Q = q_{n-1}$, $0 < P' = p_n \leq Q' = q_n$ are well-known (they follow as a result of the $RCF \rightarrow OCF$ algorithm or can be directly deduced from $p_{n-1}q_n - p_nq_{n-1} = \pm 1$). Properties (*) and (**) follow from (2.4), (2.5), and from Lemma 3.3.

(ii) \implies (i) Consider the partition

$$(g, \infty) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3, \quad \text{where}$$

$$\mathcal{S}_1 = (g, 1) \cup (2 + g, 3) \cup (4 + g, 5) \cup \dots,$$

$$\mathcal{S}_2 = [1, 2) \cup [3, 4) \cup [5, 6) \cup \dots,$$

$$\mathcal{S}_3 = [2, 2 + g) \cup [4, 4 + g) \cup [6, 6 + g) \cup \dots$$

For each matrix $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}_O$ with $\lambda = \lambda_M$, define

$$k_M = \begin{cases} 2\ell - 1 & \text{if } \lambda \in \mathcal{S}_2, [\lambda] = 2\ell - 1, \ell \geq 1, \\ 2\ell + 1 & \text{if } \lambda \in \mathcal{S}_1, [\lambda] = 2\ell, \ell \geq 0, \text{ and } \{\lambda\} > g, \\ 2\ell - 1 & \text{if } \lambda \in \mathcal{S}_3, [\lambda] = 2\ell, \ell \geq 1, \text{ and } \{\lambda\} < g. \end{cases}$$

Note that

$$k_M \geq 3 \iff \lambda > 2 + g = G^2.$$

We prove the following statement:

Lemma 3.5. *Let $x \in \Omega$ and $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}_O$ with $\tilde{Q} = \min\{Q, Q'\} > 1$ and satisfying (*) or (**). There exist $e_M \in \{\pm 1\}$ and $M_0 = \begin{pmatrix} P_0 & P \\ Q_0 & Q \end{pmatrix} \in \mathcal{R}_O$ such that*

$$M = M_0 \begin{pmatrix} 0 & e_M \\ 1 & k_M \end{pmatrix}, \quad (3.5)$$

$e_M + k_{M_0} \geq 2$, M_0 satisfies the corresponding property () or (**), and $\tilde{Q}_0 = \min\{Q_0, Q\} \leq \tilde{Q}$. Furthermore, if $\lambda = \lambda_M \in \mathcal{S}_1 \cup \mathcal{S}_2$, then we can take $\tilde{Q}_0 < \tilde{Q}$.*

Proof of Lemma 3.5. Consider the following integers:

$$e_M = \begin{cases} 1 & \text{if } \lambda \in \mathcal{S}_2 \cup \mathcal{S}_3, \\ -1 & \text{if } \lambda \in \mathcal{S}_1. \end{cases}$$

$$\begin{aligned}
Q_0 &= \begin{cases} Q' - (2\ell - 1)Q & \text{if } \lambda \in \mathcal{S}_3, [\lambda] = 2\ell, \ell \geq 1, \text{ and } \{\lambda\} < g, \\ Q' - (2\ell - 1)Q & \text{if } \lambda \in \mathcal{S}_2, [\lambda] = 2\ell - 1, \ell \geq 1, \\ (2\ell + 1)Q - Q' & \text{if } \lambda \in \mathcal{S}_1, [\lambda] = 2\ell, \ell \geq 0, \text{ and } \{\lambda\} > g. \end{cases} \\
&= \begin{cases} (1 + \{\lambda\})Q & \text{if } \lambda \in \mathcal{S}_3, \\ \{\lambda\}Q & \text{if } \lambda \in \mathcal{S}_2, \\ (1 - \{\lambda\})Q & \text{if } \lambda \in \mathcal{S}_1. \end{cases} \\
P_0 &= \begin{cases} P' - (2\ell - 1)P & \text{if } \lambda \in \mathcal{S}_3, [\lambda] = 2\ell, \ell \geq 1, \text{ and } \{\lambda\} < g, \\ P' - (2\ell - 1)P & \text{if } \lambda \in \mathcal{S}_2, [\lambda] = 2\ell - 1, \ell \geq 1, \\ (2\ell + 1)P - P' & \text{if } \lambda \in \mathcal{S}_1, [\lambda] = 2\ell, \ell \geq 0, \text{ and } \{\lambda\} > g. \end{cases}
\end{aligned}$$

Equality (3.5) holds in all cases with this choice for Q_0 and P_0 . One plainly checks that

$$\lambda_0 := \frac{Q}{Q_0} \in \begin{cases} (2 + g, \infty) & \text{if } \lambda \in \mathcal{S}_1, \\ (1, \infty) & \text{if } \lambda \in \mathcal{S}_2, \\ (g, 1] & \text{if } \lambda \in \mathcal{S}_3. \end{cases}$$

In particular this shows that $\lambda_0 > g$. The inequality $e_M + k_{M_0} \geq 2$ is trivial when $\lambda \in \mathcal{S}_2 \cup \mathcal{S}_3$. When $\lambda \in \mathcal{S}_1$ we have $\lambda_0 > 2 + g$ hence $k_{M_0} \geq 3$ and $e_M + k_{M_0} \geq 2$.

Clearly $\begin{pmatrix} 0 & e_M \\ 1 & k_M \end{pmatrix} \equiv A \pmod{2}$. The inequalities $0 \leq P_0 \leq Q_0$ follow immediately from $P_0Q - PQ_0 = \pm 1$ and $P < Q$, the latter one being a consequence of the assumption $\tilde{Q} > 1$. The fact that M_0 satisfies either (*) or (**) follows from Lemma 3.6. \square

Back to the proof of Proposition 3.4, note that when $\lambda \in (g, 1]$ one has $0 < Q_0 = Q - Q' < Q' < Q$ (the first inequality holds because $G < 2$), while for $\lambda \in (\mathcal{S}_1 \cup \mathcal{S}_2) \setminus (g, 1)$ it is plain that $0 < Q_0 < Q < Q'$. Hence whenever $\lambda \in \mathcal{S}_1 \cup \mathcal{S}_2$ one has $\min\{Q_0, Q\} < \min\{Q, Q'\}$.

When $\lambda \in \mathcal{S}_3$ one only has $\min\{Q_0, Q\} = \min\{Q, Q'\}$ (actually $Q < Q_0 < Q'$). However, in this case $e_M = -1$ so $k_{M_0} \geq 3$, and $\lambda_{M_0} = \frac{Q}{Q_0} \in (g, 1)$. Thus one can apply the same procedure to M_0 and find $M_{-1} = \begin{pmatrix} P_{-1} & P_0 \\ Q_{-1} & Q_0 \end{pmatrix} \in \mathcal{R}_0$ that satisfies (*) or (**), and such that $M_0 = M_{-1} \begin{pmatrix} 0 & e_{M_0} \\ 1 & k_{M_0} \end{pmatrix}$, $e_{M_0} + k_{M_{-1}} \geq 2$, and $\tilde{Q}_{-1} := \min\{Q_{-1}, Q_0\} < \tilde{Q}_0 = \tilde{Q}$ (this inequality is strict because $\lambda_0 \in (g, 1) \subseteq \mathcal{S}_1$).

We next discuss the case $\tilde{Q} = 1$. When $Q' = 1 \leq Q$, the inequality $\frac{Q'}{Q} = \frac{1}{Q} \geq g$ yields $Q = 1$. Hence $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, with $\frac{0}{1}, \frac{1}{1}$ successive convergents of every $x \in (0, 1)$ that satisfies $0 < \frac{x-1}{-x} < 1$, i.e. of every $x \in (\frac{1}{2}, 1)$. Suppose now $Q = 1 < Q'$. When $\frac{1}{G} < \frac{Q'}{Q} = Q' < 2 + g$, one has $Q' = 2$ and only the matrices $M = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ may arise. But the former matrix is not admissible being $\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2}$, while the latter matrix corresponds to $0 < \frac{2x-1}{-x+1} < 1$, hence $x \in (\frac{1}{2}, \frac{2}{3})$, $e_1 = 1$ and $a_1 = [\frac{1}{x}] = 1$, and indeed $\frac{1}{1}, \frac{1}{2}$ are successive convergents in $OCF(x)$ for every $x \in (\frac{1}{2}, \frac{2}{3})$. When $2 + g < \frac{Q'}{Q} = Q'$ the only matrices that may arise are $M = \begin{pmatrix} 0 & 1 \\ 1 & Q' \end{pmatrix}$ with $Q' \geq 3$ odd, and respectively $M = \begin{pmatrix} 1 & Q'-1 \\ 1 & Q' \end{pmatrix}$ with $Q' \geq 4$ even. The inequality for the former is $0 < |\frac{Q'x-1}{-x}| < 1$, which gives $x \in (\frac{1}{Q'+1}, \frac{1}{Q'}) \cup (\frac{1}{Q'}, \frac{1}{Q'-1})$ with Q' odd, so that $a_1 = Q'$ (and $e_1 = 1$ respectively $e_1 = -1$). The inequality for the latter is $0 < |\frac{Q'x-Q'+1}{-x+1}| < 1$, giving $\frac{Q'}{Q'+1} > x > \frac{Q'-2}{Q'-1} \geq \frac{2}{3}$ so $e_1 = 1$, $a_1 = 1$. Furthermore one has $\frac{1}{Q'} < \frac{1}{x} - 1 = T_D(x) < \frac{1}{Q'-2}$ with $Q' - 1 \geq 3$ odd integer, so $a_2 = Q' - 1$ and

$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & Q'_{-1} \end{pmatrix}$, showing that indeed $\frac{1}{1}, \frac{Q'-1}{Q'}$ are successive convergents in $OCF(x)$ for every x with $\frac{Q'}{Q'+1} > x > \frac{Q'-2}{Q'-1}$ and $Q' \geq 4$ even.

This inductive process on \tilde{Q} now implies that (3.4) holds for some $e_1, \dots, e_{n-1} \in \{\pm 1\}$ and a_1, \dots, a_n odd positive integers with $e_i + a_i \geq 2, \forall i \in \{1, \dots, n-1\}$. Conditions (*) and (**) show that x lies between $\frac{p_n - p_{n-1}}{q_n - q_{n-1}}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ when $q_n > q_{n-1}$, and between $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ when $q_n < q_{n-1}$. So x is of the form $[(a_1, e_1), (a_2, e_2), \dots, (a_{n-1}, e_{n-1}), (a_n + t, *)]$ for some $t \in (-1, 1)$ when $q_n > q_{n-1}$, and respectively $t \in (0, 1)$ when $q_n < q_{n-1}$. Therefore $\frac{p_{n-1}}{q_{n-1}} = \frac{P}{Q}, \frac{p_n}{q_n} = \frac{P'}{Q'}$ are successive convergents of x . \square

Lemma 3.6. *With the definitions from the proof of implication (ii) \implies (i) in Proposition 3.4 one has*

- (i) *If $g < \lambda < 1$ then $0 < E_M(x) < 1 \implies |E_{M_0}(x)| < 1$.*
- (ii) *If $1 \leq \lambda < 2 + g$, then $0 < E_M(x) < 1 \implies 0 < E_{M_0}(x) < 1$.*
- (iii) *If $2\ell + g < \lambda < 2\ell + 1, \ell \geq 1$, then $|E_M(x)| < 1 \implies -1 < E_{M_0}(x) < 0$.*
- (iv) *If $2\ell - 1 \leq \lambda < 2\ell + g, \ell \geq 2$, then $|E_M(x)| < 1 \implies 0 < E_{M_0}(x) < 1$.*

Proof. In all cases $0 < E_M(x) = \frac{Q'x - P'}{-Qx + P} < 1$ is equivalent with x lying between $\frac{P'}{Q'}$ and $\frac{P'+P}{Q'+Q}$, while $0 < E_{M_0}(x) = \frac{Qx - P}{-Q_0x + P_0} < 1$ is equivalent to x lying between $\frac{P}{Q}$ and $\frac{P+P_0}{Q+Q_0}$.

(i) In this case $Q_0 = Q - Q' < Q$ and $-1 < E_{M_0}(x) < 1$ is equivalent to x lying between $\frac{P+P_0}{Q+Q_0} = \frac{2P-P'}{2Q-Q'}$ and $\frac{P-P_0}{Q-Q_0} = \frac{P'}{Q'}$. The conclusion follows because $\frac{2P-P'}{2Q-Q'} < \frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'}$ when $\frac{P}{Q} < \frac{P'}{Q'}$ and $\frac{P'}{Q'} < \frac{P'+P}{Q'+Q} < \frac{P}{Q} < \frac{2P-P'}{2Q-Q'}$ when $\frac{P'}{Q'} < \frac{P}{Q}$.

(ii) In this case $\frac{P+P_0}{Q+Q_0} = \frac{P'}{Q'}$ and x between $\frac{P'}{Q'}$ and $\frac{P'+P}{Q'+Q}$ implies x between $\frac{P}{Q}$ and $\frac{P'}{Q'}$.

(iii) In this case $0 < Q_0 = (2\ell + 1)Q - Q' < Q < Q', -1 < E_M(x) < 1$ is equivalent to x lying between $\frac{P'+P}{Q'+Q}$ and $\frac{P'-P}{Q'-Q}$, and $-1 < E_{M_0}(x) < 0$ is equivalent to x lying between $\frac{P}{Q}$ and $\frac{P-P_0}{Q-Q_0} = \frac{P'-2\ell P}{Q'-2\ell Q}$. The implication follows because either $\frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} < \frac{P'-P}{Q'-Q} < \frac{P'-2\ell P}{Q'-2\ell Q}$ or $\frac{P'-2\ell P}{Q'-2\ell Q} < \frac{P'-P}{Q'-Q} < \frac{P'}{Q'} < \frac{P'+P}{Q'+Q} < \frac{P}{Q}$.

(iv) In this case $Q' > Q$ and $\frac{P+P_0}{Q+Q_0} = \frac{P'-(2\ell-2)P}{Q'-(2\ell-2)Q}$. The implication follows because $-1 < E_M(x) < 1$ is equivalent with x lying between $\frac{P'+P}{Q'+Q}$ and $\frac{P'-P}{Q'-Q}$, $0 < E_{M_0}(x) < 1$ is equivalent with x lying between $\frac{P}{Q}$ and $\frac{P+P_0}{Q+Q_0}$, and either $\frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} < \frac{P'-P}{Q'-Q} < \frac{P'-(2\ell-2)P}{Q'-(2\ell-2)Q}$ or $\frac{P'-(2\ell-2)P}{Q'-(2\ell-2)Q} < \frac{P'-P}{Q'-Q} < \frac{P'}{Q'} < \frac{P'+P}{Q'+Q} < \frac{P}{Q}$ holds. \square

The following statement will also be useful:

Lemma 3.7. *Denominators of successive convergents in OCF satisfy*

- (i) $q_{n+2} > q_n$.
- (ii) $q_{n+3} > q_n$.
- (iii) $q_{n+2} > \min\{q_n, q_{n+1}\}$.

Proof. By Proposition 3.4 and its proof $\frac{q_{n+2}}{q_{n+1}} > 2 \implies \frac{q_{n+2}}{q_n} > 2g > 1, \frac{q_{n+2}}{q_{n+1}} \in (1, 2) \implies \frac{q_{n+1}}{q_n} > 1 \implies \frac{q_{n+2}}{q_n} > 1$, and $\frac{q_{n+2}}{q_{n+1}} \in (g, 1) \implies \frac{q_{n+1}}{q_n} > 2 + g \implies \frac{q_{n+2}}{q_n} > g(2 + g) > 1$. Thus in all possible cases $q_{n+2} > q_n$, which establishes (i).

(ii) follows from $\frac{q_{n+3}}{q_{n+2}} \in (g, 1) \implies \frac{q_{n+2}}{q_{n+1}} > 2 + g \implies \frac{q_{n+3}}{q_n} > (2 + g)g^2 = 1, \frac{q_{n+3}}{q_{n+2}} = \lambda \in (1, 2) \implies \frac{q_{n+2}}{q_{n+1}} = \frac{1}{\lambda-1} \implies \frac{q_{n+3}}{q_n} > \frac{\lambda}{\lambda-1}g > 2g > 1, \frac{q_{n+3}}{q_{n+2}} \in (2, 2 + g) \implies \frac{q_{n+2}}{q_{n+1}} \in (g, 1) \implies \frac{q_{n+1}}{q_n} > 2 + g \implies \frac{q_{n+3}}{q_n} > 2g(2 + g) > 1$, and $\frac{q_{n+3}}{q_n} > 2 + g \implies \frac{q_{n+3}}{q_n} > (2 + g)g^2 = 1$.

To prove (iii) suppose that $q_{n+2} \leq q_{n+1}$. Then $\frac{q_{n+2}}{q_{n+1}} \in (g, 1)$, which gives in turn $\frac{q_{n+1}}{q_n} > 2 + g$, and therefore $\frac{q_{n+2}}{q_n} > g(2 + g) > 1$. \square

Remark. Proposition 3.2 was originally proved, using a different method, by Kraaikamp and Lopes [7], but Proposition 3.4 is, to the best of our research, new. Our proofs have an additional benefit of implying how to derive a_n and e_{n-1} (and hence q_{n-2}) if only q_{n-1} and q_n are known.

Our investigations yielded yet another method of proof, significantly longer but more direct, which we sketch here. Examples 1.8 in [8] explain how to algorithmically generate the *OCF* expansion of x from the *RCF* expansion of x using insertion

$$\begin{aligned} & \text{(replacing } [[\dots, (a_n, 1), (a_{n+1}, e_{n+1}), \dots]] \\ & \text{with } [[\dots, (a_n + 1, -1), (1, 1), (a_{n+1} - 1, e_{n+2}), \dots]]) \end{aligned}$$

and singularization

$$\begin{aligned} & \text{(replacing } [[\dots, (a_n, e_n), (1, 1), (a_{n+2}, e_{n+2}), \dots]] \\ & \text{with } [[\dots, (a_n + e_n, -e_n), (a_{n+2} + 1, e_{n+2}), \dots]]) \end{aligned}$$

Both of these operations alter the sequence of convergents: insertion adds a new convergent, while singularization deletes one. Nevertheless, it can be shown that if $\frac{P}{Q}, \frac{P'}{Q'}$ are successive *RCF* convergents to some x , then either $\frac{P}{Q}, \frac{P'}{Q'}$ are successive *OCF* convergents to x or $\frac{Q-P}{Q}, \frac{Q'-P'}{Q'}$ are successive *OCF* convergents to $1 - x$. (Only one of these pairs forms a matrix that is congruent to I , A , or B modulo 2.) By carefully following how insertion and singularization change the last e_{n-1} and a_n in the *RCF* expansion of $\frac{P'}{Q'}$ into the last e_{m-1} and a_m of the *OCF* expansion of $\frac{P'}{Q'}$, we can determine exactly what $e(M)$ and $a(M)$ must be and hence how to derive P_0 and Q_0 . A similar proof works for the *ECF* case as well.

4. ESTIMATING THE LIMITING JOINT DISTRIBUTION FOR *ECF* AND *OCF*

For each $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}$ and $\xi \in (0, 1]$ denote by $I_\xi^+(M)$ (respectively, $I_\xi^-(M)$) the set of solutions x of $0 \leq E_M(x) \leq \xi$ (respectively, of $-\xi \leq E_M(x) \leq 0$). The Lebesgue measure of $I_\xi^\pm(M)$ is

$$f_\xi^\pm(Q, Q') = \left| \frac{P' \pm \xi P}{Q' \pm \xi Q} - \frac{P'}{Q'} \right| = \frac{\xi}{Q'(Q' \pm \xi Q)}.$$

The integral

$$\begin{aligned} F_\pm &= F_\pm(x_1, x_2, x_3, x_4) := \int_{R/x_2}^\infty dv \int_0^{\min\{x_3 v, x_1 R\}} du f_{x_4}^\pm(u, v) \\ &= \pm \int_{R/x_2}^\infty \frac{dv}{v} \log \left| \frac{v \pm x_4 \min\{x_3 v, x_1 R\}}{v} \right| \\ &= \pm \int_{x_3/x_2}^\infty \frac{dw}{w} \log \left| \frac{w \pm x_3 x_4 \min\{w, x_1\}}{w} \right| \end{aligned}$$

can be expressed when $x_3 \geq x_1 x_2$ as

$$F_\pm = \pm \int_0^{x_1 x_2 x_4} \frac{dt}{t} \log(1 \pm t) = \mp \text{Li}_2(\mp x_1 x_2 x_4),$$

and when $x_3 < x_1x_2$ as

$$\begin{aligned} F_{\pm} &= \int_{x_3/x_2}^{x_1} \frac{dw}{w} \log(1 \pm x_3x_4) \pm \int_{x_1}^{\infty} \frac{dw}{w} \log \frac{w \pm x_1x_3x_4}{w} \\ &= \pm \log(1 \pm x_3x_4) \log \frac{x_1x_2}{x_3} \mp \text{Li}_2(\mp x_3x_4), \end{aligned}$$

so F_{\pm} is as in (1.5).

4.1. The ECF case. By Lemma 3.1 and Proposition 3.2, for each $R > 1$ and $x \in \Omega$ there is a unique $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}_E$ with $Q \leq R < Q'$ and $|E_M(x)| < 1$. Given $x_1, x_2, x_3, x_4 \in (0, 1)$ consider $\mathcal{N}_{x_1, x_2, x_3, x_4}^{E, \pm}(x, R)$, the number of matrices $M \in \mathcal{R}_E$ that satisfy (1.1) and (1.2). One has

$$\mathcal{L}^{E, \pm}(R) = \mathcal{L}_{x_1, x_2, x_3, x_4}^{E, \pm}(R) = \int_0^1 \mathcal{N}_{x_1, x_2, x_3, x_4}^{E, \pm}(x, R) dx.$$

For $\Gamma \in \{I, J, A, B\}$ we shall estimate

$$\mathcal{L}_{\Gamma}^{\pm}(R) := \sum_{\substack{M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{R}_E \\ Q' \geq R/x_2 \\ Q \leq \min\{x_3Q', x_1R\} \\ M \equiv \Gamma \pmod{2}}} f_{x_4}^{\pm}(Q, Q').$$

This can be done by Möbius summation, as in the following standard lemmas (for Lemma 4.2 see, e.g., [2, Lemma 2.1]).

Lemma 4.1. *For every interval J , every function $g \in C^1(J)$ of total variation $T_J g$, and every integer x , with σ_0 the divisor counting function,*

$$\sum_{\substack{a \in J, b \in [1, q] \\ ab \equiv x \pmod{q} \\ (a, q) = 1}} g(a) = \sum_{\substack{a \in J \\ (a, q) = 1}} g(a) = \frac{\varphi(q)}{q} \int_J g(u) du + O(\sigma_0(q)(\|g\|_{\infty} + T_J g)).$$

Lemma 4.2. *For every interval J , every $V \in C^1[0, N]$, and every $\ell \in \mathbb{N}$*

$$\sum_{\substack{1 \leq q \leq N \\ (q, \ell) = 1}} \frac{\varphi(q)}{q} V(q) = C(\ell) \int_0^N V(u) du + O_{\ell}(\|V\|_{\infty} + T_0^N V \log N),$$

with

$$C(\ell) = \frac{1}{\zeta(2)} \prod_{\substack{p \in \mathcal{P} \\ p \nmid \ell}} \left(1 + \frac{1}{p}\right)^{-1}.$$

Changing b to $q - b$ in Lemma 4.1, we infer

Corollary 4.3. *Suppose q is an odd positive integer. For every interval J , every $g \in C^1(J)$, and every integer x*

$$\sum_{\substack{a \in J, b \in [1, q/2] \\ ab \equiv x \text{ or } -x \pmod{q} \\ (a, q) = 1}} g(a) = \frac{\varphi(q)}{q} \int_J g(u) du + O(\|g\|_{\infty} + T_J g \sigma_0(q)).$$

Since $P'Q - PQ' = \pm 1$, P', Q even and Q' odd entail P odd, we infer (with $Q = 2q$, $P' = 2p'$, \bar{x} the multiplicative inverse of $x \pmod{Q'}$)

$$\begin{aligned}
\mathcal{L}_I^\pm(R) &= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 1 \pmod{2}}} \sum_{\substack{q \in [1, \min\{x_3 Q', y_1 R\}/2] \\ p' \in [1, Q'/2] \\ p'q \equiv \pm 1 \pmod{Q'}}} f_{x_4}^\pm(2q, Q') \\
&= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 1 \pmod{2}}} \left(\frac{\varphi(Q')}{Q'} \int_0^{\min\{x_3 Q', x_1 R\}/2} f_{x_4}^\pm(2q, Q') dq + O_\varepsilon(Q'^{-2+\varepsilon}) \right) \\
&= \frac{1}{2} \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 1 \pmod{2}}} \frac{\varphi(Q')}{Q'} \int_0^{\min\{x_3 Q', x_1 R\}} f_{x_4}^\pm(u, Q') du + O_\varepsilon(R^{-1+\varepsilon}) \\
&= \frac{C(2)F_\pm}{2} + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}).
\end{aligned} \tag{4.1}$$

On the other hand, $P'Q - PQ' = \pm 1$ and Q' even entail that both Q and P' are odd, and the condition P even is equivalent to $P'Q \equiv \pm 1 \pmod{2Q'}$. Since in this case $\varphi(2Q') = 2\varphi(Q')$, we infer

$$\begin{aligned}
\mathcal{L}_J^\pm(R) &= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 0 \pmod{2}}} \sum_{\substack{Q \in [1, \min\{x_3 Q', x_1 R\}] \\ P' \in [1, Q'] \\ P'Q \equiv \pm 1 \pmod{2Q'}}} f_{x_4}^\pm(Q, Q') \\
&= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 0 \pmod{2}}} \left(\frac{\varphi(2Q')}{2Q'} \int_0^{\min\{x_3 Q', x_1 R\}} f_{x_4}^\pm(u, Q') du + O_\varepsilon(Q'^{-2+\varepsilon}) \right) \\
&= \left(\frac{1}{\zeta(2)} - C(2) \right) F_\pm + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}),
\end{aligned}$$

leading to

$$\mathcal{L}^{E,\pm}(R) = \mathcal{L}_I^\pm(R) + \mathcal{L}_J^\pm(R) = \frac{2F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}),$$

and concluding the proof of (1.3).

The corresponding estimates for $\mathcal{L}_B^\pm(R)$ and $\mathcal{L}_A^\pm(R)$ are useful for the *OCF* situation. To estimate $\mathcal{L}_B^\pm(R)$, note that $P'Q - PQ' = \pm 1$ and Q' even entail that both P' and Q are odd,

$\varphi(2Q') = 2\varphi(Q')$, and thus

$$\begin{aligned}
\mathcal{L}_B^\pm(R) &= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 0 \pmod{2}}} \sum_{\substack{Q \in [1, \min\{x_3 Q', x_1 R\}] \\ P' \in [1, Q'], P'Q \equiv \pm 1 \pmod{Q'} \\ \frac{P'Q \mp 1}{Q'} \equiv 1 \pmod{2}}} f_{x_4}^\pm(Q, Q') \\
&= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 0 \pmod{2}}} \left(\sum_{\substack{Q \in [1, \min\{x_3 Q', x_1 R\}] \\ P' \in [1, Q'], P'Q \equiv \pm 1 \pmod{Q'}}} f_{x_4}^\pm(Q, Q') - \sum_{\substack{Q \in [1, \min\{x_3 Q', x_1 R\}] \\ P' \in [1, Q'], P'Q \equiv \pm 1 \pmod{2Q'}}} f_{x_4}^\pm(Q, Q') \right) \\
&= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 0 \pmod{2}}} \left(\left(\frac{2\varphi(Q')}{Q'} - \frac{\varphi(2Q')}{2Q'} \right) \int_0^{\min\{x_3 Q', x_1 R\}} f_{x_4}^\pm(u, Q') du + O_\varepsilon(Q'^{-2+\varepsilon}) \right) \quad (4.2) \\
&= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 0 \pmod{2}}} \left(\frac{\varphi(Q')}{Q'} \int_0^{\min\{x_3 Q', x_1 R\}} f_{x_4}^\pm(u, Q') du + O_\varepsilon(Q'^{-2+\varepsilon}) \right) \\
&= \left(\frac{1}{\zeta(2)} - C(2) \right) F_\pm + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}).
\end{aligned}$$

Finally, $P'Q - PQ' = \pm 1$ and P even entail that both P' and Q are odd, and so

$$\begin{aligned}
\mathcal{L}_A^\pm(R) &= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 1 \pmod{2}}} \sum_{\substack{Q \in [1, \min\{x_3 Q', x_1 R\}] \\ P' \in [1, Q'], P'Q \equiv \pm 1 \pmod{2Q'}}} f_{x_4}^\pm(Q, Q') \\
&= \sum_{\substack{Q' \geq R/x_2 \\ Q' \equiv 1 \pmod{2}}} \left(\frac{\varphi(2Q')}{2Q'} \int_0^{\min\{x_3 Q', x_1 R\}} f_{x_4}^\pm(u, Q') du + O_\varepsilon(Q'^{-2+\varepsilon}) \right) \quad (4.3) \\
&= \frac{C(2)}{2} F_\pm + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}).
\end{aligned}$$

4.2. The OCF case. This requires more caution as the sequence of denominators of successive convergents is not monotonically increasing in general. We wish to characterize those matrices $M \in \mathcal{R}_O$ for which $\frac{P}{Q}, \frac{P'}{Q'}$ are successive convergents of $x \in \Omega$ and $Q = q_{n_R} \leq R < Q' = q_{n_R+1}$. A priori, Lemma 3.7 shows that for each $R > 1$ there is at least one pair and at most two pairs (Q, Q') of denominators of successive convergents of x with $Q \leq R < Q'$. Moreover, if there are two such pairs (Q, Q') , then they must be of the form (q_{n_R}, q_{n_R+1}) or (q_{n_R+2}, q_{n_R+3}) . We wish to precisely distinguish n_R from $n_R + 2$. Because all predecessors of Q_0 in the sequence of denominators of OCF convergents are $< Q$ by Lemma 3.7, equality $(Q, Q') = (q_{n_R}, q_{n_R+1})$ occurs exactly when

$$Q \leq R < Q' \quad \text{and} \quad R > Q_0.$$

Note that if $\lambda = \frac{Q'}{Q} \in \mathcal{S}_1 \cup \mathcal{S}_2$, then necessarily $Q > Q_0$. Furthermore, if $\lambda \in \mathcal{S}_3$, then $Q < Q_0$. The contribution of those pairs (Q, Q') with $\lambda \in \mathcal{S}_3$ and $Q_0 = Q(1 + \{\lambda\}) > R$ should be subtracted, and so we can write

$$\begin{aligned}
\mathcal{L}^{O,+}(R) &= \mathcal{L}_I^+(R) + \mathcal{L}_A^+(R) + \mathcal{L}_B^+(R) - \mathcal{D}_1(R), \\
\mathcal{L}^{O,-}(R) &= \mathcal{L}_I^-(R) + \mathcal{L}_A^-(R) + \mathcal{L}_B^-(R) - \mathcal{D}_2(R) - \mathcal{D}_3(R),
\end{aligned}$$

with

$$\mathcal{D}_1(R) = \sum_{\substack{M \in \mathcal{R}_O, Q' > R/x_2 \\ Q \leq \min\{x_3 Q', x_1 R\} \\ \lambda = Q'/Q \in \mathcal{S}_3, Q(1+\{\lambda\}) > R}} f_{x_4}^+(Q, Q') = \sum_{\ell \geq 1} \sum_{\substack{M \in \mathcal{R}_O, Q' > R/x_2 \\ Q \leq \min\{x_3 Q', x_1 R\} \\ 2\ell Q \leq Q' < (2\ell+g)Q \\ Q' > R+(2\ell-1)Q}} \frac{x_4}{Q'(Q' + x_4 Q)},$$

$$\mathcal{D}_2(R) = \sum_{\substack{M \in \mathcal{R}_O, Q' > R/x_2 \\ Q \leq \min\{x_3 Q', x_1 R\} \\ \lambda = Q'/Q \in [2, 2+g), Q' > R+Q}} \frac{x_4}{Q'(Q' - x_4 Q)},$$

$$\mathcal{D}_3(R) = \sum_{\substack{M \in \mathcal{R}_O, Q' > R/x_2 \\ Q \leq \min\{x_3 Q', x_1 R\} \\ \lambda = Q'/Q \in \mathcal{S}_3, \lambda > G^2 \\ Q(1+\{\lambda\}) > R}} f_{x_4}^-(Q, Q') = \sum_{\ell \geq 2} \sum_{\substack{M \in \mathcal{R}_O, Q' > R/x_2 \\ Q \leq \min\{x_3 Q', x_1 R\} \\ 2\ell Q \leq Q' < (2\ell+g)Q \\ Q' > R+(2\ell-1)Q}} \frac{x_4}{Q'(Q' - x_4 Q)}.$$

Clearly $\mathcal{D}_2(R) = 0$ when $\min\{x_1 x_2, x_3\} \leq g^2$. When $\min\{x_1 x_2, x_3\} > g^2$, the method employed in (4.1), (4.2) and (4.3) applies, leading to

$$\mathcal{D}_2(R) = \frac{D_2(x_1, x_2, x_3, x_4)}{\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}),$$

with D_2 as in (1.6).

The estimation of $\mathcal{D}_1(R)$ is slightly more involved because ℓ can take infinitely many values. Note that $\mathcal{D}_1(R) = 0$ unless $\min\{x_1 x_2, x_3\} > \frac{1}{2\ell+g}$. For each $\ell \in \mathbb{N}$ consider the integral

$$I_\ell^+(R) := \iint_{\substack{v \geq R/x_2, u \leq \min\{x_3 v, x_1 R\} \\ 2\ell u \leq v \leq (2\ell+g)u \\ v > R+(2\ell-1)u}} \frac{x_4 du dv}{v(v + x_4 u)}.$$

The change of variables $(v, u) = (Ry, Rx)$ shows that $I_\ell^+(R)$ does not depend on R and is given by (1.7). Note also that

$$I_\ell^+(R) \leq \int_0^{x_1} dx \int_{2\ell x}^{(2\ell+1)x} \frac{dy}{y^2} \ll \frac{1}{\ell^2}. \quad (4.4)$$

A trivial estimate yields

$$\begin{aligned} \sum_{\substack{\ell \geq R^{1/2} \\ R/x_2 \leq Q' \leq (2\ell+1)R}} \sum_{\substack{Q' \\ \frac{Q'}{2\ell+1} \leq Q \leq \frac{Q'}{2\ell}}} \frac{1}{Q'(Q' + x_4 Q)} &\leq \sum_{\substack{\ell \geq R^{1/2} \\ 1 \leq Q' \leq (2\ell+1)R}} \sum_{\substack{Q' \\ \frac{Q'}{2\ell+1} \leq Q \leq \frac{Q'}{2\ell}}} \frac{1}{Q^2 \ell^2} \\ &\ll \sum_{\ell \geq R^{1/2}} \frac{1}{\ell^2} \sum_{Q \in [1, 2R]} \sum_{Q' \in [2\ell Q, (2\ell+1)Q]} \frac{1}{Q^2} \ll \frac{\log R}{R^{1/2}}, \end{aligned}$$

and thus in the definition of $\mathcal{D}_1(R)$ we may take $\ell \in [1, R^{1/2}]$ inserting an error term $\ll R^{-1/2} \log R$. Employing Lemma 4.1, the resulting main term can be expressed as

$$\begin{aligned} & \sum_{\ell \leq R^{1/2}} \sum_{\substack{Q' \geq R/x_2 \\ Q' < (2\ell+g)x_1 R}} \left(\frac{\varphi(Q')}{Q'} \int_{\frac{Q'}{2\ell+g}}^{\min\{\frac{Q'}{2\ell}, x_3 Q', x_1 R, \frac{Q'-R}{2\ell-1}\}} \frac{x_4 du}{Q'(Q' + x_4 u)} + O(Q'^{-2+\varepsilon}) \right) \\ &= \left(\sum_{\ell \leq R^{1/2}} \sum_{\substack{Q' \geq R/x_2 \\ Q' < (2\ell+g)x_1 R}} \frac{\varphi(Q')}{Q'} \int_{\frac{Q'}{2\ell+g}}^{\min\{\frac{Q'}{2\ell}, x_3 Q', x_1 R, \frac{Q'-R}{2\ell-1}\}} \frac{x_4 du}{Q'(Q' + x_4 u)} \right) + O_\varepsilon(R^{-1/2+\varepsilon}). \end{aligned}$$

Employing now Lemma 4.2, the main term above becomes

$$\sum_{\ell \leq R^{1/2}} \left(\frac{I_\ell^+}{\zeta(2)} + O\left(\frac{\log R}{R^{1/2}}\right) \right),$$

and so

$$\mathcal{D}_1(R) = \frac{1}{\zeta(2)} \sum_{\ell \leq R^{1/2}} I_\ell^+ + O_\varepsilon(R^{-1/2+\varepsilon}). \quad (4.5)$$

From (4.5) and (4.4) we eventually infer

$$\mathcal{D}_1(R) = \frac{1}{\zeta(2)} \sum_{\ell \geq 1} I_\ell^+ + O_\varepsilon(R^{-1/2+\varepsilon}).$$

The sum $\mathcal{D}_3(R)$ is similarly estimated as in formulas (1.6) and (1.7).

5. JOINT DISTRIBUTION FOR NAKADA'S α -EXPANSIONS

We illustrate how explicit renewal type results can be obtained in the case of Nakada's α -expansions NCF_α , $\alpha \in [\frac{1}{2}, 1]$. Such continued fractions, defined in [10], have been studied in [10, 6]. Here the unit interval is replaced by $\Omega_\alpha = [\alpha - 1, \alpha)$ and the Gauss shift by the map $T_\alpha : \Omega_\alpha \rightarrow \Omega_\alpha$ defined for $x \neq 0$ by³

$$T_\alpha(x) = \left\lfloor \frac{1}{x} \right\rfloor - \left[\left\lfloor \frac{1}{x} \right\rfloor + 1 - \alpha \right].$$

A construction of the natural extension \overline{T}_α on a space $\underline{\Omega}_\alpha \subset \mathbb{R}^2$, together with an explicit invariant Borel probability measure μ_α on $\underline{\Omega}_\alpha$ was found by Nakada [10]. He also proved that $(\underline{\Omega}_\alpha, \overline{T}_\alpha, \mu_\alpha)$ is a Kolmogorov automorphism. With $g = \frac{1}{G} = 1 - g^2$ the set $\underline{\Omega}_\alpha$ is given for $g < \alpha \leq 1$ by

$$[\alpha - 1, (1 - \alpha)/\alpha] \times [0, 1/2] \cup ((1 - \alpha)/\alpha, \alpha) \times [0, 1] \cup [\alpha - 1, 0) \times \{1/2\},$$

and for $\frac{1}{2} \leq \alpha \leq g$ by

$$\begin{aligned} & [\alpha - 1, (1 - 2\alpha)/\alpha] \times [0, g^2] \cup ((1 - 2\alpha)/\alpha, (2\alpha - 1)/(1 - \alpha)] \times [0, 1/2] \\ & \cup ((2\alpha - 1)/(1 - \alpha), \alpha) \times [0, g] \cup [-g^2, (1 - 2\alpha)/\alpha] \times \{g^2\} \\ & \cup ((1 - 2\alpha)/\alpha, 0) \times \{1/2\}. \end{aligned}$$

³Here we use the notation from Sections 5 and 6 of [6].

Kraaikamp's thoughtful analysis (see especially Theorem (5.3) and Definitions (5.7) and (5.8) of [6]) also provides characterizations of pairs of successive convergents for such continued fractions if $\alpha \in [\frac{1}{2}, 1]$.

Proposition 5.1. *For each $x \in \Omega_\alpha \setminus \mathbb{Q}$ the following are equivalent:*

- (i) $\frac{P}{Q}, \frac{P'}{Q'}$ successive convergents in $NCF_\alpha(x)$ with $Q, Q' > 0$.
- (ii) $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in GL_2(\mathbb{Z})$ and $(E_M(x), 1/\lambda_M) \in \underline{\Omega}_\alpha$.

This dynamical system was studied by Kraaikamp [6] in the more general setting of S -expansions, and the above proposition can be likewise generalized if we replace $NCF_\alpha(x)$ with $CF_S(x)$, the S -expansion of x , and replace $\underline{\Omega}_\alpha$ with $\underline{\Omega}_S$, the space of the natural extension associated to S .

We wish to estimate the Lebesgue measure $\mathcal{L}_{x_1, x_2, x_3, x_4}^{(\alpha), \pm}(R)$ of the set of numbers $x \in \Omega_\alpha \setminus \mathbb{Q}$ for which there exist $\frac{P}{Q}, \frac{P'}{Q'}$ successive convergents in $NCF_\alpha(x)$ that satisfy (1.1) and (1.2). We shall require that x_1, x_2, x_3 are in the set $(0, 1]$ if $g < \alpha \leq 1$, in the set $(0, 1/2]$ if $\alpha = g$, and in the set $(0, g]$ if $1/2 \leq \alpha < g$; moreover, we require $x_4 \in (0, \alpha]$ when we look at \mathcal{L}^+ and $x_4 \in (0, 1 - \alpha]$ when we look at \mathcal{L}^- . The set $\underline{\Omega}_\alpha$ is a union of rectangles and horizontal line segments, but we may ignore the line segments for large R : in particular, the inequality $Q' \geq \frac{R}{x_2}$ shows that the pair $(Q', Q) = (2, 1)$ makes no contribution to \mathcal{L}^\pm for $R > 2$, so the situation $\lambda_M^{-1} = \frac{1}{2}$ can be ignored, and λ_M is always rational, so the situation $\lambda_M^{-1} = g^2$ can also be ignored. As a result, the cases that appear in $\mathcal{L}_{x_1, x_2, x_3, x_4}^{(\alpha), \pm}(R)$ for $R > 2$ are exactly the following:

$$\begin{aligned} \text{For } g < \alpha \leq 1: & \begin{cases} \lambda_M = Q'/Q > 2 \text{ and } \alpha - 1 \leq E_M(x) < \alpha, & \text{or} \\ 1 \leq \lambda_M < 2 \text{ and } \frac{1-\alpha}{\alpha} < E_M(x) < \alpha. \end{cases} \\ \text{For } 1/2 \leq \alpha \leq g: & \begin{cases} \lambda_M > G^2 \text{ and } \alpha - 1 \leq E_M(x) < \alpha, & \text{or} \\ 2 < \lambda_M < G^2 \text{ and } \frac{1-2\alpha}{\alpha} < E_M(x) < \alpha, & \text{or} \\ G < \lambda_M < 2 \text{ and } \frac{2\alpha-1}{1-\alpha} < E_M(x) < \alpha. \end{cases} \end{aligned}$$

The varying lower bounds on λ_M depending on the value of α are the reason for our case-based restrictions on the values of x_1, x_2, x_3 .

Let $\mathcal{L}_{x_1, x_2, x_3, x_4}^+(\alpha; R)$ denote the Lebesgue measure of the set of numbers $x \in [0, 1] \setminus \mathbb{Q}$ for which there exists $M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in GL_2(\mathbb{Z})$ with $Q, Q' > 0$, $\frac{P}{Q}, \frac{P'}{Q'} \in [\alpha - 1, \alpha)$ and (1.1) together with $0 \leq \frac{Q'x - P'}{-Qx + P} \leq x_4$ hold. The corresponding set where the latter inequality is replaced by $-x_4 \leq \frac{Q'x - P'}{-Qx + P} \leq 0$ is denoted by $\mathcal{L}_{x_1, x_2, x_3, x_4}^-(\alpha; R)$. In both cases, x_1, x_2, x_3, x_4 are parameters in $(0, 1]$. When $\alpha = 1$, it is clear that \mathcal{L}^+ is exactly the joint distribution considered in [16] (where the notation used is $N(R)$). However, by the following equation

$$\begin{aligned} \mathcal{L}_{x_1, x_2, x_3, x_4}^\pm(\alpha; R) &= \sum_{Q' \geq R/x_2} \sum_{\substack{Q \in (0, \min\{x_3 Q', x_1 R\}] \\ P' \in (\alpha-1)Q' + [0, Q') \\ P'Q \equiv \pm 1 \pmod{Q'}}} \frac{x_4}{Q'(Q' \pm x_4 Q)} \\ &= 2 \sum_{Q' \geq R/x_2} \sum_{\substack{Q \in (0, \min\{x_3 Q', x_1 R\}] \\ (Q, Q')=1}} \frac{x_4}{Q'(Q' \pm x_4 Q)} = \mathcal{L}_{x_1, x_2, x_3, x_4}^\pm(R), \end{aligned}$$

we see that $\mathcal{L}^\pm(\alpha; R)$ does not depend on α . As R tends to infinity, \mathcal{L}^\pm converges to $2F^\pm/\zeta(2)$.

The joint distributions $\mathcal{L}^{(\alpha),\pm}$ and \mathcal{L}^\pm can now be directly related as below. For the sake of space and readability we omit the appearance of x_1 , x_2 , and R , which are assumed to be the same on the left- and right-hand sides of the equations.

When $g < \alpha \leq 1$, we have

$$\mathcal{L}_{x_3, x_4}^{(\alpha), +} = \begin{cases} \mathcal{L}_{\min\{x_3, 1/2\}, x_4}^+ & \text{if } 0 \leq x_4 \leq \frac{1-\alpha}{\alpha}, \\ \mathcal{L}_{x_3, x_4}^+ - \mathcal{L}_{x_3, (1-\alpha)/\alpha}^+ + \mathcal{L}_{\min\{x_3, 1/2\}, (1-\alpha)/\alpha}^+ & \text{if } \frac{1-\alpha}{\alpha} \leq x_4 < \alpha, \end{cases}$$

$$\mathcal{L}_{x_3, x_4}^{(\alpha), -} = \mathcal{L}_{\min\{x_3, 1/2\}, x_4}^- \quad \text{if } 0 \leq x_4 \leq 1 - \alpha.$$

When $\frac{1}{2} \leq \alpha \leq g$, we have

$$\mathcal{L}_{x_3, x_4}^{(\alpha), +} = \begin{cases} \mathcal{L}_{\min\{x_3, 1/2\}, x_4}^+ & \text{if } 0 \leq x_4 \leq \frac{2\alpha-1}{1-\alpha}, \\ \mathcal{L}_{x_3, x_4}^+ - \mathcal{L}_{x_3, (2\alpha-1)/(1-\alpha)}^+ + \mathcal{L}_{\min\{x_3, 1/2\}, (2\alpha-1)/(1-\alpha)}^+ & \text{if } \frac{2\alpha-1}{1-\alpha} \leq x_4 < \alpha. \end{cases}$$

$$\mathcal{L}_{x_3, x_4}^{(\alpha), -} = \begin{cases} \mathcal{L}_{\min\{x_3, 1/2\}, x_4}^- & \text{if } 0 \leq x_4 \leq \frac{2\alpha-1}{\alpha}, \\ \mathcal{L}_{\min\{x_3, g^2\}, x_4}^- + \mathcal{L}_{\min\{x_3, 1/2\}, (2\alpha-1)/\alpha}^- - \mathcal{L}_{\min\{x_3, g^2\}, (2\alpha-1)/\alpha}^- & \text{if } \frac{2\alpha-1}{\alpha} \leq x_4 \leq 1 - \alpha. \end{cases}$$

Recall that $x_3 \leq g$ in this case.

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REFERENCES

- [1] W. W. Adams, *On a relationship between the convergents of the nearest integer and regular continued fractions*, Math. Comp. **33** (1979), 1321–1331.
- [2] F. P. Boca, R. N. Gologan, *On the distribution of the free path length of the linear flow in a honeycomb*, Ann. Inst. Fourier (Grenoble) **69** (2009), 1043–1075.
- [3] F. Cellarosi, *Renewal-type theorem for continued fractions with even partial quotients*, Ergodic Theory Dynam. Systems **29** (2009), 1451–1478.
- [4] F. Cellarosi, *Limiting curlicue measures for theta sums*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), 466–497.
- [5] K. Dajani, D. Hensley, C. Kraaikamp, V. Masarotto, *Arithmetic and ergodic properties of ‘flipped’ continued fraction algorithms*, Acta Arith. **153** (2012), 51–79.
- [6] C. Kraaikamp, *A new class of continued fraction expansions*, Acta Arith. **57** (1991), 1–39.
- [7] C. Kraaikamp, A. O. Lopes, *The theta group and the continued fraction expansion with even partial quotients*, Geom. Dedicata **59** (1996), 293–333.
- [8] V. Masarotto, *Metric and arithmetic properties of a new class of continued fraction expansions*, Master Thesis, Università di Padova and Leiden University, 2008/2009.
- [9] B. Minnigerode, *Über eine neue Methode, die Pellsche Gleichung aufzulösen*, Nachr. Göttingen 1873.
- [10] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426.

- [11] G. J. Rieger, *On the metrical theory of continued fractions with odd partial quotients*, in: Colloquia Mathematica Societatis János Bolyai 34, Topics in Classical Number Theory, Budapest, 1981, pp. 1371–1418.
- [12] F. Schweiger, *Continued fractions with odd and even partial quotients*, Arbeitsberichte Math. Institut Universität Salzburg **4** (1982), 59–70; *On the approximation by continued fractions with odd and even partial quotients*, Arbeitsberichte Math. Institut Universität Salzburg **1-2** (1984), 105–114.
- [13] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford University Press, Oxford, 1995.
- [14] Ya. G. Sinai, *Limit theorem for trigonometric sums. Theory of curlicues*, Russian Math. Surveys **63** (2008), 1023–1029.
- [15] Y. G. Sinai, C. Ulcigrai, *Renewal-type limit theorem for the Gauss map and continued fractions*, Ergodic Theory Dynam. Systems **28** (2008), 643–655.
- [16] A. V. Ustinov, *On the statistical properties of elements of continued fractions*, Doklady Mathematics **79** (2009), No.1, 87–89.
- [17] A. V. Ustinov, *The mean value number of steps in the Euclidean algorithm with least value remainders*, Mat. Zametki **85** (2009), 153–156.
- [18] H. C. Williams, *Some results concerning the nearest integer continued fraction expansion of \sqrt{D}* , J. Reine Angew. Math. **315** (1980), 1–15.

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